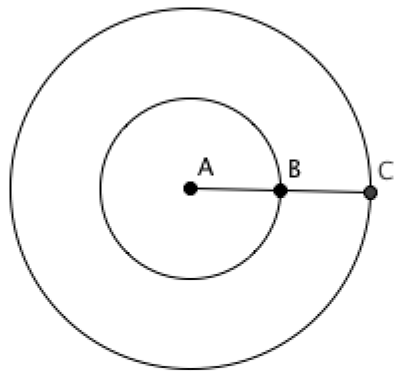
1st Annual Lexington Mathematical Tournament
Individual Round

Solutions

1. Answer: $\boxed{6}$
Solution: The only pairs of even positive integers summing to 8 are 2 and 6, or two 4's. However, since the integers have to be distinct, we take 2 and 6, and the larger is 6.

2. Answer: $\boxed{3\pi}$



Solution: C_2 is a circle with radius 2, and C_1 is a circle with radius 1 lying entirely inside. The region in question is C_2, with C_1 taken out, so its area is the area of C_2 minus the area of C_1. This is $(2)^2\pi - (1)^2\pi = 3\pi$.

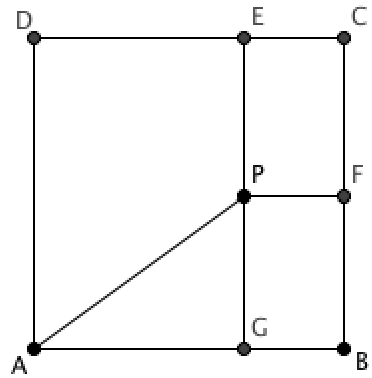
3. Answer: $\boxed{-2}$
Solution: Say the starting integer is x. Doubling gives $2x$, subtracting 4 gives $2x - 4$, and halving gives $x - 2$. We now want to subtract the original integer, so we get $(x - 2) - x = -2$.

4. Answer: $\boxed{1}$
Solution: Say some integer x was a divisor of all three. Then, if we were to prime factorize x, so that a prime p was a divisor of x (p may be equal to x itself), p would be a divisor of all three as well. However, A, B, C are all written in their prime factorizations, and we see that no prime number is a divisor of all 3. Thus, x cannot have any prime divisors, meaning $x = 1$, and 1 is the largest positive divisor of all three integers.

5. Answer: $\boxed{1}$
Solution: Let $x = 2010$. The quantity is equal to $x^2 - (x-1)(x+1) = x^2 - (x^2 - 1) = 1$.

6. Answer: $\boxed{4/7}$
Solution: Al can get a red and a blue in two ways: he can draw red first, then blue, or blue first, then red. In the first case, he draws red with probability $3/7$, then blue with probability $4/6$, since there are 6 marbles left and 4 are blue. The probability of these two events is $(3 \cdot 4)/(7 \cdot 6) = 12/42$. Then, in the second case, he draws blue with probability $4/7$, then red with probability $3/6$, since there are 6 marbles left and 3 are red, and again, the probability is $12/42$. Adding gives our answer of $24/42 = 4/7$.

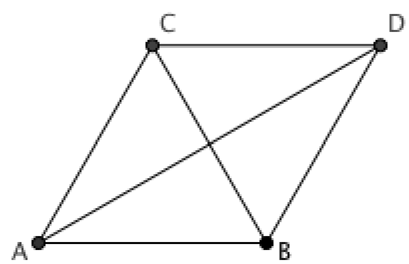
7. Answer: $\boxed{5}$



Solution: Let the perpendiculars from P to sides DC, CB, BA have feet at E, F, G, respectively. We are given that $PE = 3$ and $PF = 2$. Thus, $PG = 6 - 3 = 3$ and $AG = 6 - 2 = 4$. Applying the Pythagorean Theorem to triangle APG, we get $AP = \sqrt{3^2 + 4^2} = 5$.

8. Answer: $\boxed{61}$
Solution: It's hard to count the elements now, but if we make some modifications, they'll become easier to count. First, we'll add 82 to everything, so that our list starts at 3, to give $\{3, 6, \dots, 183\}$. Now, these are all multiples of 3, so we can divide everything by 3 to give $\{1, 2, \dots, 61\}$. Both operations preserve the number of elements of the set, and clearly, there are 61 elements in the third set.

9. Answer: $\boxed{\sqrt{3}/4}$



Solution: Note that triangles ABD and ACD are congruent, since $AC = CD = DB = BA$, and they share side AD (in other words, we have SSS congruence). Thus, they have the same area, which is half the area of the quadrilateral $ABDC$. This, in turn, is equal to the area of equilateral triangle ABC, since ABC and DBC also have the same area. We can use the formula $\sqrt{3}s^2/4$ for the area of an equilateral triangle of side length s, to get the answer $\sqrt{3}/4$ (since it's to know where formulas come from, for a hint as to why this is the case, draw a perpendicular AE to BC, where E is on BC. What kind of triangle is AEB?).

10. Answer: $\boxed{15}$
Solution: 2502 is a little bit more than $2500 = 50^2$. So the square numbers less than 2500 are $1^2, 2^2, 3^2, \dots, 50^2$. To count the number of these that are squares of prime numbers, we only need the number of primes between 1 and 50, inclusive. From here, we just count that there are 15: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.

11. Answer: $\boxed{49}$
Solution: We can use the classic formula $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ (to get this, imagine instead adding $(1+n) + (2+(n-1)) + \cdots + ((n-1)+2) + (n+1)$, which adds two copies of the sum, then dividing by 2 so we end up with just one copy). Now, we want to divide this by n and not get an

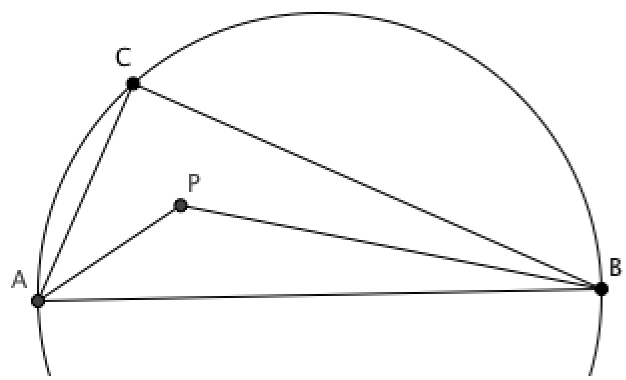
integer: the result will be $\dfrac{n+1}{2}$, which is an integer if and only if n is odd, so it's not an integer if and only if n is even. There are 49 even integers less than 100 $(2, 4, \ldots, 98)$, so this is our answer.

12. Answer: $\boxed{12}$
Solution: Ted can only be certain of Tim's integer if, of the integers between 2 and 15, inclusive, Tim's integer is the only one with its number of factors. For example, if Tim's integer were 2, Ted could not deduce this only from the fact that 2 has 2 factors, since, for example, 3 also has exactly two factors. Now, we can make the following list:

2 factors: 2, 3, 5, 7, 11, 13 (prime numbers)
3 factors: 4, 9
4 factors: 6, 8, 10, 14, 15
6 factors: 12

12 is the only integer in this interval that has a unique number of factors, so Tim's integer must be 12.

13. Answer: $\boxed{135}$



Solution: Because AB is a diameter, $\angle ACB = 90°$, since $\angle ACB$ subtends 180 degree arc AB. Let $\angle CAB = x$. This makes $\angle CBA = 90° - x$, since the sum of the measures of the angles in the triangle must be 180°. Now, because PA and PB bisect their respective angles, $\angle PAB = x/2$ and $\angle PBA = (90° - x)/2$. We can use the fact that the sum of the measures of the angles in triangle PAB is 180 degrees, to find that $\angle APB = 180° - x/2 - (90° - x)/2 = 135°$. So in fact, $\angle APB$ is always the same, that is, equal to 135 degrees (so accordingly, the maximum is 135).

During the competition, a protest, asserting that 225 was also a possible answer to this problem because $\angle APB$ can also be interpreted as a reflex angle, was accepted. Obviously, given the original answer, this was not the intent of the problem, but because arriving at the answer of 225 required the same steps as in arriving at the intended answer at 135, the alternate answer was accepted.

14. Answer: $\boxed{180}$
Solution: To start, we'll want to choose 3 people for part 1, which we can do in $\binom{6}{3} = 20$ ways. Then, the remaining 3 people will be left to do part 2. In each of the two groups of 3, we have 3 ways of choosing a captain, so our answer is $3 \cdot 3 \cdot 20 = 180$.

15. Answer: $\boxed{11 - 2\sqrt{6}}$

Solution: Completing the square separately in x and y, we rewrite the equation as $(x-11)^2 + (y-8)^2 = 72$. In the xy-plane, this represents a circle with center $(11, 8)$ and radius $\sqrt{72} = 6\sqrt{2}$. The smallest possible value of x is represented in the left-most point on the circle, which must appear on a radius that is horizontal (parallel to the x-axis). Thus, the x-coordinate will be $11-6\sqrt{2}$, as we start from the center, and move to the left at a distance equal to the radius.

16. Answer $\boxed{11}$
Solution: Let the three digit integer be abc, with digits a, b, c. Algebraically, its value is $100a + 10b + c = 19(a + b + c)$. Rearranging, we get $81a = 9b + 18c$, and dividing by 9 gives $9a = b + 2c$. Now, we want to find digits a, b, c satisfying this equation. $a \neq 0$, since we want a three-digit integer. Also, the right hand side is at most 27, when $b = c = 9$, so the left hand side is at most 27, meaning a is at most 3.

If $a = 1$, we get the equation $b + 2c = 9$, which has the solutions $(9, 0), (7, 1), (5, 2), (3, 3), (1, 4)$.
If $a = 2$, we get the equation $b + 2c = 18$, which has the solutions $(8, 5), (6, 6), (4, 7), (2, 8), (0, 9)$.
If $a = 3$, we get the equation $b + 2c = 27$, which has the solution $(3, 3)$.

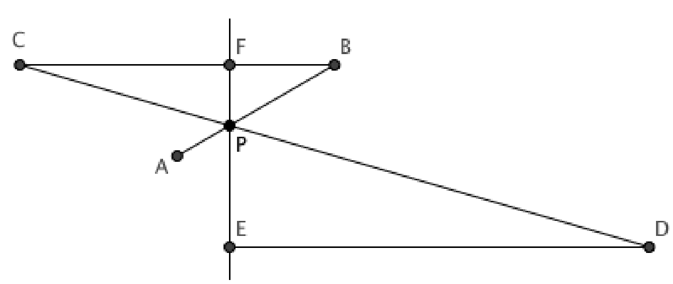
This gives us 11 solutions in total, corresponding to the 11 positive integers with the desired property.

17. Answer: $\boxed{20}$
First, consider what happens when we use both 2009 and 2011. Then, these must be painted on opposite sides of the cube. Then, the four remaining sides are adjacent to both 2009 and 2011, so we need them all to be labeled with 2010. All variations of this labeling scheme are rotations of each other, so we only have 1 possibility in this case. Otherwise, we use at most 2 numbers, either 2010 and 2009 or 2010 and 2011. For this solution we will think of painting the faces of the cube red or blue, with red representing 2010 and blue representing 2009 or 2011. For each coloring that includes blue, we multiply by 2, since we can either assign blue to 2009 or 2011.

We do casework on the number of red faces. If all six are red, we only have one coloring. Similarly if five are red, since we can always rotate to put the blue face on top. If there are four red faces, we have two colorings, since the two blue faces can either be adjacent or opposite. With three red faces, if some two red faces are opposite, then the third red face must be placed in between, and all of these configurations are the same. Otherwise, we will need all three faces to share a vertex, which can be done in one way. By symmetry (flipping red and blue), the number of colorings for 4, 5, 6 red faces are the same as the number of colorings for 2, 1, 0 red faces, respectively.

Our total number of colorings is $1 + 1 + 2 + 2 + 2 + 1 + 1 = 10$, but 9 have at least one blue face, which we multiply by 2. Our answer is thus $1 + 2 \cdot 9 + 1 = 20$.

18. Answer: $\boxed{3\sqrt{3}}$



Solution: Since $BP/AP = 2$, we get $BP = 2$. Then, if BC hits l at F, since BC is perpendicular to l, and line APB and l make a $60°$ angle, PFB is a 30-60-90 triangle, with $BF = \sqrt{3}$ and $PF = 1$. Now, $CF/BF = 2$, so $CF = 2\sqrt{3}$. Next, let the perpendicular from D to l meet l at E. Then, CFP DEP, because FC and ED are both perpendicular to l and thus parallel to each other. The ratio of similarity is $1/2$, since $CP/DP = 1/2$. Thus, $EP = 2$ and $ED = 4\sqrt{3}$.

To get the area of triangle BPD, we will find the area of trapezoid $BFED$, then subtract the areas of BFP and PED. $FBDE$ is a right trapezoid with bases $FB = \sqrt{3}$ and $DE = 4\sqrt{3}$, and height $FP + PE = 1 + 2 = 3$, so its area is $\frac{1}{2} \cdot 5\sqrt{3} \cdot 3 = \frac{15}{2} \cdot \sqrt{3}$. BFP is a right triangle with bases $1, \sqrt{3}$, so its area is $\frac{1}{2} \cdot \sqrt{3}$. Finally, right triangle PED has legs with lengths $2, 4\sqrt{3}$, so its area is $4\sqrt{3}$. Our answer is thus $\sqrt{3}(15/2 - 1/2 - 4) = 3\sqrt{3}$.

19. Answer: $\boxed{4374}$
Solution: We have $162 = 2 \cdot 3^4$ as the prime factorization of 162. Therefore, the integers that are relatively prime to 162 are those that have no prime factors of 2 or 3. Looking at remainders when divided by $2 \cdot 3 = 6$ (that is, modulo 6), integers with remainders of $0, 2, 4$ when divided by 6 are all divisible by 2, and those with remainders of $0, 3$ are divisible by 3, whereas integers with remainders $1, 5$ are divisible by neither. Thus, we want to add up the integers less than 162 with remainders of 1 or 5 when divided by 6.

Remainders of 1: $1 + 7 + 13 + \cdots + 157 = \dfrac{(1 + 157)(27)}{2} = 79 \cdot 27.$

Remainders of 5: $5 + 11 + 17 + \cdots + 161 = \dfrac{(5 + 161)(27)}{2} = 83 \cdot 27.$

Thus, the total sum is $27(79 + 83) = 27 \cdot 162 = 4374$ (can you think of a sneakier way of adding up the terms? Hint: pair up each of the integers with remainder 1 with one of remainder 5).

20. Answer: $\boxed{78}$
Solution: It's possible to compute a, b, c, d, e, f separately, by substituting $x + 1$ for x, then comparing coefficients of the two polynomials on either side (after expanding the one on the left hand side). This, however, is fairly tedious. What we can do instead is to note that plugging in special values of x gives us certain relations that may be helpful. For example, plugging in $x = 1$ gives $f = -7$, because on the right hand side, all addends become zero except f. Furthermore, plugging in $x = 2$ gives $a + b + c + d + e + f = 12$ and plugging in $x = 0$ gives $-a + b - c + d - e + f = -14$. Adding these equations and dividing by 2 makes a, c, e cancel, and what's left is $b + d + f = -1$. Plugging back into the first equation, $a + c + e = 13$.

Now, we can notice that the equation of what we want factors as $(a + c + e)(b + d)$, which we can rewrite as $(a+c+e)(b+d+f-f)$. Using what we already know, this is equal to $(13)(-1+7) = 78$.